MATH2050C Solution to Assignment 13

Supplementary Problems

1. Show that for $a > 0$, $\lim_{x\to\infty} \dfrac{x^a}{e^x} = 0$ and $\lim_{x\to-\infty} |x|^a e^x = 0$.

   Solution Fix some n satisfying $n > a$. Then for $x \geq 1$, $0 \leq \dfrac{x^a}{e^x} \leq \dfrac{x^n}{e^x}$ and by squeezing

$$0 \leq \lim_{x\to\infty} \frac{x^a}{e^x} \leq \lim_{x\to\infty} \frac{x^n}{e^x} = 0,$$

   by Theorem 13.1. On the other hand, letting $y = -x$,

$$\lim_{x\to-\infty} |x|^a e^x = \lim_{y\to\infty} \frac{y^a}{e^y} = 0 \ .$$

   Note This problem shows that the exponential function grows faster than any polynomial at infinity.

2. Show that for $a > 0$, $\lim_{x\to\infty} \dfrac{\ln x}{x^a} = 0$.

   Solution Setting $y = \ln x$, $y \to \infty$ as $x \to \infty$, so

$$\lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{y\to\infty} \frac{y}{e^{ay}} = 0 \ .$$

   Note This problem shows that the log function decays faster than any positive power at infinity.

3. Determine the domain(s) of definition and continuity of the function given by the formula $\sqrt{\dfrac{x^2-4}{x+3}}$.

   Solution $\{x : \dfrac{x^2-4}{x+3} \geq 0\} = [2, \infty)$. Hence the domain of definition and continuity is $[2, \infty)$.

4. Determine the domain(s) of definition and continuity of the function given by the formula $\mathrm{sgn}(x^2 - 1)$.

   Solution This function is well-defined everywhere, and it is continuous everywhere except at $x = 1, -1$.

5. Determine the domain(s) of definition and continuity of the function given by the formula $\ln(-x^2 + 2x + 3)$.

   Solution $-x^2 + 2x + 3 > 0$ iff $x \in (-1, 3)$, hence the domain of definition and continuity of $\ln(-x^2 + 2x + 3)$ is $(-1, 3)$.